A unified approach to q-special functions of the Laplace type

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Abstract

We propose a unified approach to q-special functions, which are degenerations of basic hypergeometric functions ${}_2\varphi_1(a, b; c; q, x)$. We obtain a list of seven different class of q-special functions: ${}_2\varphi_1$, ${}_1\varphi_1$, two different types of the q-Bessel functions, the q-Hermite-Weber functions, two different types of the q-Airy functions. We show that there exist a relation between two types of the q-Airy functions.

1 Introduction

In study of classical special functions, unified theories help us to understand special functions. We can study classical special functions defined by differential equations easily by confluence of singular points or the method of separation of variables. Since we do not have such a unified theory for q-special functions, it is difficult to decide whether two different special functions have relations or not. For example, we know three different q-Bessel functions and two different q-Airy functions and we know one relation between the first and the second q-Bessel function. But it is not easy to determine they are completely different or not in general.

Recently many textbooks on q-special functions have been published such as Koekoek-Lesky-Swarttouw[12], but they have not shown unified theories of q-special functions although they list up a huge list of special functions. In this paper, we give a unified theory for q-special functions, which come from degeneration of the basic hypergeometric functions ${}_2\varphi_1(a, b; c; q, x)$. In our viewpoint, we have essentially two different types of q-Bessel functions. We also see that two types of q-Airy equations are essentially equivalent but they are different as q-series. A connection formula of q-Airy equations is recently found by T. Morita [15]. Our list is not enough to study whole of the Askey scheme. It is a future problem to expand our unified theory to include the Askey-Wilson polynomials.

One of the most famous theory on a unified approach to classical special functions is by confluence of singularities [1, 11]. We denote irregular singular points of the Poincaré rank $r - 1$ as (r) in Figure 1. We may consider that regular singular points are singularity of the Poincaré rank 0. We remark that we consider the Poincaré rank of the Bessel equation at the infinity is $1/2$, because we think ${}_0F_1(c; x)$ is a *true* Bessel function (see section 2). The author
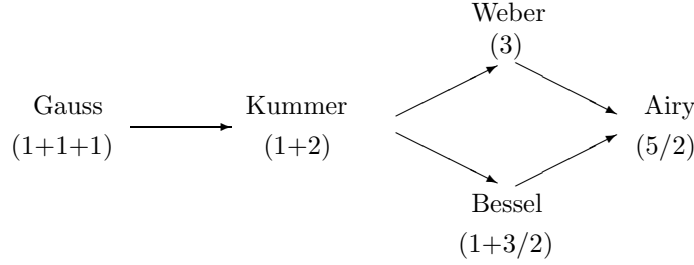
Weber
(3)

Gauss   →   Kummer   Airy

(1+1+1)   (1+2)   (5/2)

Bessel
(1+3/2)

Figure 1: The coalescent diagram of classical special functions

has learned from Professor Hideyuki Majima that the word "confluent" appeared in the second edition of *Modern Analysis* [22] and did not appear in the first edition written without Watson in 1902.

Another approach is by separation of variables of the Laplacian by orthogonal coordinates. This method is useful to study the Mathieu functions and the spheroidal wave functions [1, 14].

In the study of q-special functions, we do not have such a unified approach since it is difficult to consider confluence of singular points or separation of variables in the q-analysis.

The third approach is classify differential equations of the Laplace type [21]

$$(a_0 + b_0 x)\frac{d^2 y}{dx^2} + (a_1 + b_1 x)\frac{dy}{dx} + (a_2 + b_2 x)y = 0$$

by means of change of variables $x \to px + q$ and $y \to g(x)y$. We obtain Kummer, Weber, Bessel and Airy functions from equations of the Laplace type. We review the third approach in section 2.

In section 3, we review different types of q-Bessel functions and q-Airy functions. In this section we introduce two important tools to study q-difference equations. One is a shearing transformation and the second is gauge transformations by q-products and theta functions.

The third approach can be easily modified in q-difference equations. We call a q-difference equation of the second order with the linear coefficients

$$(a_0 + b_0 x)u(xq^2) + (a_1 + b_1 x)u(xq) + (a_2 + b_2 x)u(x) = 0$$

the hypergeometric type. In section 4, we classify q-difference equations of the hypergeometric type and obtain seven types of q-difference equations:

In section 5, we compare our classification of q-difference equations of the hypergeometric type to special solutions of the q-Painlevé equations.

We may consider q-difference equations in the matrix form $Y(xq) = A(x)Y(x)$. The matrix form has more parameters than a single equation of the higher order. These parameters are redundant but we can reduce the number of parameters since the matrix form of q-difference equation have more transformations than the scalar form of higher order. For a q-difference equation of the hypergeomet-
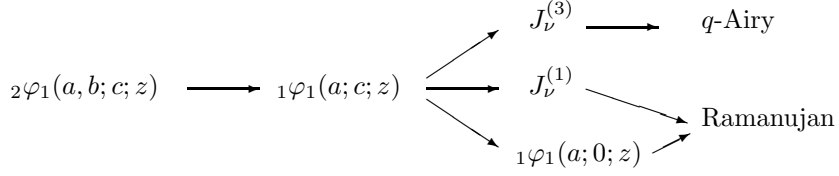
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$$
\begin{array}{ccc}
& & J_\nu^{(3)} \longrightarrow \quad q\text{-Airy} \\
_2\varphi_1(a,b;c;z) \longrightarrow \ _1\varphi_1(a;c;z) & \nearrow \\
& \searrow & J_\nu^{(1)} \searrow \\
& & _1\varphi_1(a;0;z) \nearrow \quad \text{Ramanujan}
\end{array}
$$

Figure 2: The coalescent diagram of q-special functions

ric type, the corresponding matrix form is a 2×2 system

$$Y(xq) = (A_0 + A_1 x)Y(x).$$

In section 6 we classify this type of equations.

2 Differential equation of the Laplace type

In this section, we show confluent hypergeometric series and classification of second order differential equations of the Laplace type. Although these results are already known, we review classical results[21] in order to fix our notations.

2.1 Confluent hypergeometric series

We set confluent hypergeometric series found by Kummer.

$$
\begin{aligned}
_1F_1(a;c;z) &= \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n-1)} \frac{x^n}{n!}, \\
_0F_1(-;c;z) &= \sum_{n=0}^{\infty} \frac{1}{c(c+1)\cdots(c+n-1)} \frac{x^n}{n!}.
\end{aligned}
$$

Kummer's second confluent hypergeometric series $_0F_1(-;c;z)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2} + n\frac{dy}{dx} - y = 0.$$

The infinity is an irregular singular point of the Poincaré rank $1/2$. The author does not know standard term (*degenerate confluent hypergeometric series* or *confluent limit hypergeometric series*) for $_0F_1$. Tricomi [21] used the notation $E_\nu(z) = {}_0F_1(-;\nu+1;-z)/\Gamma(\nu+1)$, Jahnke and Emde [7] used another notation $\Lambda_\nu(x) = {}_0F_1(-;\nu+1;-x^2/4)$.

It is known that the Bessel functions are represented by confluent hypergeometric series in two ways. One is related to $_1F_1$:

$$J_\nu(z) = \frac{z^\nu e^{-iz}}{2^\nu \Gamma(\nu+1)} {}_1F_1(1/2+\nu, 1+2\nu; 2iz),$$

and the second is related to $_0F_1$:

$$J_\nu(2\sqrt{z}) = \frac{z^{\nu/2}}{\Gamma(\nu+1)} {}_0F_1(-; \nu+1; -z).$$

It is convenient to take $_0F_1(-;c;z)$ as a standard form of the Bessel function at least theoretically. But in many applications in mathematical physics, $J_\nu(z)$ is used since it is obtained as the Cylindrical functions.

2.2 Classification of equations of the Laplace type

A second order differential equation of the Laplace type

$$(a_0 + b_0 x)\frac{d^2 y}{dx^2} + (a_1 + b_1 x)\frac{dy}{dx} + (a_2 + b_2 x)y = 0 \tag{1}$$

is solved by special functions explained in subsection 2.1. We remark that by the transformations (λ, μ and h are constants)

$$x = \lambda\xi + \mu, \quad y = e^{hx}\eta,$$

$\eta(\xi)$ also satisfy a differential equations of the Laplace type (1).

We set

$$A(h) = a_0 h^2 + a_1 h + a_2, \quad B(h) = b_0 h^2 + b_1 h + b_2.$$

Classification of second order differential equations of the Laplace type is as follows (see Capitolo Primo in Tricomi's book[21]).

Theorem 2.1. *The equation* (1) *is solved by special functions* $_1F_1(a;c;z)$, $J_\nu(z)$, $_1F_1(a;1/2;z)$ *and* $J_{1/3}(z)$.

(i) *Assume that* $b_0 \neq 0$ *and* $\Delta := b_1^2 - 4b_0 b_2 \neq 0$. *We set*

$$h = \frac{-b_1 \pm \sqrt{\Delta}}{2b_0}, \ \lambda = -\frac{b_0}{B'(h)}, \ \mu = -\frac{a_0}{b_0}, \ a = \frac{A(h)}{B'(h)}, \ c = \frac{a_1 b_0 - a_0 b_1}{b_0^2}.$$

Then (1) *is solved by* $\eta = {}_1F_1(a;c;\xi)$, *where* $x = \lambda\xi + \mu$ *and* $y = e^{hx}\eta$.

(ii) *Assume that* $b_0 \neq 0$ *and* $\Delta = 0$. *We set*

$$h = -\frac{b_1}{2b_0}, \ \lambda = b_0, \ \mu = -\frac{a_0}{b_0}, \ \alpha = \frac{1}{2} - \frac{A'(h)}{2b_0}, \ \beta = 2\sqrt{A(h)}.$$

Then (1) *is solved by* $\eta = \xi^\alpha J_{2\alpha}(\beta\sqrt{\xi})$, *where* $x = \lambda\xi + \mu$ *and* $y = e^{hx}\eta$.

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(iii) Assume that $b_0 = 0$ and $a_0 b_1 \neq 0$. We set

$$h = -\frac{b_2}{b_1}, \ \ \mu = -\frac{A'(h)}{b_1}, \ \ a = \frac{A(h)}{2b_1}, \ \ k = -\frac{b_1}{2a_0}.$$

Then (1) is solved by $\eta = {}_1F_1(a; 1/2; k\xi^2)$, where $x = \xi + \mu$ and $y = e^{hx}\eta$.
(iv) Assume that $b_0 = b_1 = 0$ and $a_0 b_2 \neq 0$. We set

$$h = -\frac{a_1}{2a_0}, \ \ \mu = \frac{4a_0 a_2 - a_1^2}{4a_0 b_2}, \ \ k = \frac{2}{3}\sqrt{\frac{b_2}{a_0}}.$$

Then (1) is solved by $\eta = \sqrt{\xi} J_{1/3}(k\xi^{3/2})$, where $x = \xi + \mu$ and $y = e^{hx}\eta$.
(v) For other cases, (1) is solved by elementary functions.

3 q-Hypergeometric functions

3.1   Basic notations

We fix the notation of basic hypergeometric series. The q-Pochhammer symbol is given by

$$
\begin{aligned}
(a;q)_0 &= 1, \\
(a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad (n=1,2,3,...) \\
(a;q)_\infty &= \prod_{k=0}^{\infty}(1-aq^{k-1}), \\
(a_1,\ldots,a_r;q)_n &= \prod_{i=1}^{n}(a_i;q)_n.
\end{aligned}
$$

The basic hypergeometric series is defined by

$$
{}_r\varphi_s\left(a_1,\ldots,a_r; b_1,\ldots,b_s; q,z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-s} z^n.
$$

The theta function is defined by

$$
\theta_q(x) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2}x^n = (q, -x, -q/x; q)_\infty,
$$

and $\theta_q(x)$ satisfies a q-difference equation $x\theta_q(xq) = \theta_q(x)$.

The theta function $\theta_q(x)$ and the q-Pochhammer symbol $(a;q)_\infty$ are used to transform q-difference equations. The following simple lemma plays an important role in this paper.

Lemma 3.1. *We take a q-difference equation*

$$a(x)u(xq^2) + b(x)u(qx) + c(x)u(x) = 0,$$

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(1) We set $u(x) = v(x)/(sx;q)_\infty$. Then $v(x)$ satisfies the q-difference equation

$$(1 - sqx)a(x)v(xq^2) + b(x)v(xq) + \frac{c(x)}{1 - sx}v(x) = 0.$$

(2) We set $u(x) = \theta_q(rx)w(x)$. Then $w(x)$ satisfies the q-difference equation

$$\frac{a(x)}{rq}w(xq^2) + \frac{x}{r}b(x)w(xq) + rx^2c(x)w(x) = 0.$$

We show two examples:

(a) For

$$a(x)u(xq^2) + b(x)u(xq) + (c + dx)u(x) = 0,$$

we set $u(x) = v(x)/(-dx/c;q)_\infty$. Then we have

$$\frac{1}{c}(c - dqx)a(x)v(xq^2) + b(x)v(xq) + cv(x) = 0.$$

(b) We set $u(x) = \theta_q(ax)w(x)$ for

$$axu(xq^2) + b(x)u(xq) + cu(x) = 0.$$

Then we have

$$w(xq^2) + b(x)w(xq) + acxw(x) = 0.$$

3.2   Shearing transformations

In the study of differential equations, shearing transformations are useful to study irregular singular points whose Poincaré rank is non-integer.

Shearing transformations are also useful for q-differential equations when a slope of the Newton diagram is non-integer.

Definition 3.1. *For a q-difference equation*

$$a(x)u(xq^2) + b(x)u(xq) + c(x)u(x) = 0,$$

a shearing transformation is the following transformation:

$$x = t^2, \ \ p = \sqrt{q}, \ \ v(t) = u(x).$$

The shearing transform of the q-difference equation is given by

$$a(t^2)v(tp^2) + b(t^2)v(tp) + c(t^2)v(t) = 0.$$

We will show an example of a shearing transformation. We take a q-linear equation

$$(ax + b)u(xq^2) + cu(xq) + du(x) = 0. \tag{2}$$

Here $a \neq 0, b, c, d \neq 0$ are constants. The Newton diagram of (2) at $x = \infty$ has a slope $1/2$. The shearing transform of (2) is

$$(at^2 + b)v(tp^2) + cv(tp) + dv(t) = 0.$$

If $b \neq 0$, we set $w(t) = (p^{-1}\sqrt{-b/a}; p)_\infty v(t)$. Then $w(t)$ satisfies

$$a(t + \sqrt{-b/a})w(tp^2) + cw(tp) + d(t - p^{-1}\sqrt{-b/a})w(t) = 0.$$

If $b = 0$, we set $w(t) = \theta_p(t)v(t)$, as shown in subsection 3.1. Then $w(t)$ satisfies

$$atw(tp^2) + cw(tp) + dtw(t) = 0.$$

In any case, $w(t)$ has a regular singularity at $t = \infty$ if $ad \neq 0$.

3.3 q-Bessel functions and q-Airy functions

It is known that there exist three types of q-Bessel functions and two types of q-Airy functions. It is not clear that relations between such functions. It is known that $J_\nu^{(1)}(x; q)$ and $J_\nu^{(2)}(x; q)$ are essentially equivalent and $J_\nu^{(3)}(x; q)$ is essentially different. We show that two types of q-Airy functions are related by a shearing transformation and $\mathrm{Ai}_q(z)$ is a special case of $J_\nu^{(3)}(x; q)$.

1) q-Bessel functions:
It is known that there exist three types of q-Bessel functions $J_\nu^{(1)}(x; q)$, $J_\nu^{(2)}(x; q)$ and $J_\nu^{(3)}(x; q)$. In most of literatures, $J_\nu^{(1)}(x; q)$ and $J_\nu^{(2)}(x; q)$ are called Jackson's first and second q-Bessel functions, and $J_\nu^{(3)}(x; q)$ is called Hahn-Exton's q-Bessel function.

The three q-Bessel functions are defined as

$$J_\nu^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu \; {}_2\varphi_1\left(0, 0; q^{\nu+1}; q, -\frac{x^2}{4}\right),$$

$$J_\nu^{(2)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu \; {}_0\varphi_1\left(-; q^{\nu+1}; q, -\frac{q^{\nu+1}x^2}{4}\right),$$

$$J_\nu^{(3)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \; {}_1\varphi_1\left(0; q^{\nu+1}; q, qx^2\right).$$

The three q-Bessel functions satisfy the following q-difference equations:

$$J_\nu^{(1)} : \quad u(xq) - (q^{\nu/2} + q^{-\nu/2})u(xq^{1/2}) + \left(1 + \frac{x^2}{4}\right)u(x) = 0,$$

$$J_\nu^{(2)} : \quad \left(1 + \frac{qx^2}{4}\right)u(xq) - (q^{\nu/2} + q^{-\nu/2})u(xq^{1/2}) + u(x) = 0,$$

$$J_\nu^{(3)} : \quad u(xq) + [-(q^{\nu/2} + q^{-\nu/2}) + q^{\nu/2+1}x^2]u(xq^{1/2}) + u(x) = 0.$$

It is convenient to take the inverse shearing transform of equations above. We set $p = q^{1/2}$, $t = x^2$ and $v(t) = u(x)$. Then we have

$$J_\nu'^{(1)} : \quad v(tp^2) - (p^\nu + p^{-\nu})v(tp) + \left(1 + \frac{t}{4}\right)v(t) = 0,$$

$$J_\nu'^{(2)} : \quad \left(1 + \frac{p^2 t}{4}\right)v(tp^2) - (p^\nu + p^{-\nu})v(tp) + v(t) = 0,$$

$$J_\nu'^{(3)} : \quad v(tp^2) + [-(p^\nu + p^{-\nu}) + p^{\nu+2}t]v(tp) + v(t) = 0.$$

In the following we set

$$
\begin{aligned}
E_\nu^{(1)}(x;p) &= t^\nu{}_2\varphi_1\left(0,0;p^{2\nu+1};p,-x/4\right), \\
E_\nu^{(2)}(x;p) &= t^\nu{}_0\varphi_1\left(-;p^{2\nu+1};p,-xp^{2\nu+2}/4\right), \\
E_\nu^{(3)}(x;p) &= t^\nu{}_1\varphi_1\left(0;p^{2\nu+1};p,xp^2/4\right).
\end{aligned}
$$

$E_\nu^{(1)}(x;p)$ and $E_{-\nu}^{(1)}(x;p)$ are solutions of the equation $J_\nu'^{(1)}$. $E_\nu^{(2)}(x;p)$ and $E_{-\nu}^{(2)}(x;p)$ are solutions of the equation $J_\nu'^{(2)}$. $E_\nu^{(3)}(x;p)$ and $E_{-\nu}^{(3)}(xp^{-2\nu};p)$ are solutions of the equation $J_\nu'^{(3)}$. We use the notation $E_\nu^{(j)}(x;p)$ since Tricomi used $E_\nu(x)$ in his study in the Bessel functions (see subsection 2.1).

By Hahn's formula

$$
J_\nu^{(2)}(x;q) = (-\frac{x^2}{4};q)_\infty \cdot J_\nu^{(1)}(x;q), \quad E_\nu^{(2)}(x;p) = (-\frac{t}{4};p^2)_\infty E_\nu^{(1)}(x;p) \quad (3)
$$

In this sense $J_\nu^{(1)}$ and $J_\nu^{(2)}$ are equivalent. Fitouhi and Hamza [2] have defined another q-Bessel function $j_\alpha(x,q^2)$, which is essentially equivalent to $J_\nu^{(3)}(x;q)$.

2) q-Airy functions
It is known that there exist two different types of the q-Airy functions. The q-Airy function $\mathrm{Ai}_q(x)$ is found in the study of special solutions of the second q-Painlevé equation [9]. The limit $q \to 1$ tends to the Airy function at least around the infinity [5]. The second function $A_q(x)$, is called the *Ramanujan function*, which is found by Ismail in the study of asymptotic behavior of the q-Hermite polynomial [6].

$$
\begin{aligned}
\mathrm{Ai}_q(x) &= {}_1\varphi_1(0;-q;q,-x), \\
A_q(x) &= {}_0\varphi_1(-;0;q,-qx).
\end{aligned}
$$

It was not known any relation between the q-Airy function $\mathrm{Ai}_q(x)$ and the Ramanujan function $A_q(x)$, but recently Morita has found a connection formula between $\mathrm{Ai}_q(x)$ and $A_q(x)$ (see (8) in subsection 4.2).

The Airy function is a special case of the modified Bessel function:

$$
\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\,K_{1/3}\left(\frac{2}{3}x^{3/2}\right).
$$

As the same as the differential case, the q-Airy function $\mathrm{Ai}_q(x)$ is related to $J_\nu^{(3)}(x;q)$.

Lemma 3.2. *If $q^\nu = -1$, we have*

$$
J_\nu^{(3)}(x;q) = \frac{(-q;q)_\infty}{(q;q)_\infty}x^\nu\,\mathrm{Ai}_q(-qx^2).
$$

The q-Airy function $\mathrm{Ai}_q(x)$ satisfies a q-difference equation

$$u(xq^2) + xu(xq) - u(x) = 0, \qquad (4)$$

and $\mathrm{A}_q(x)$ satisfies a q-difference equation

$$qxu(xq^2) - u(xq) + u(x) = 0. \qquad (5)$$

These two equation connected by shearing transformation as we has seen in subsection 3.2.

Lemma 3.3. *If $u(x)$ satisfies (4),*

$$v(x) = \theta_q(-q^2 x)\mathrm{Ai}_q(1/x)$$

satisfies an inverse shearing transform of a modified equation of (4):

$$-q^5 x^2 v(xq^2) - v(xq) + v(x) = 0,$$

which is is solved by $\mathrm{A}_{q^2}(x^2 q^3)$.

We can check out the lemma above directly.

4   Classification

We classify a q-difference equation of the hypergeometric type:

$$(a_0 + b_0 x)u(xq^2) + (a_1 + b_1 x)u(xq) + (a_2 + b_2 x)u(x) = 0. \qquad (6)$$

The equation above has transformations which keep the hypergeometric type. Such transformations ware known by Hahn [4].

4.1   A q-analogue of the Riemann scheme

Before we list up transformations which keep the hypergeometric type, we introduce Matsumoto's q-analogue of the Riemann scheme [13].

Definition 4.1. *For (6), we set two characteristic polynomials $a_0\mu^2 + a_1\mu + a_2 = 0$ and $b_0 + b_1\lambda + b_2\lambda^2 = 0$ The first one is a characteristic polynomial around $x = 0$ and the second one is a characteristic polynomial around $x = \infty$. The roots of both polynomials are called characteristic exponents $\mu_1, \mu_2, \lambda_1, \lambda_2$. The characteristic exponents is considered as ∞ when $a_0 = 0$ or $b_2 = 0$. We set $\rho_1 = -a_0/b_0, \quad \rho_2 = -b_2/a_2$, which are virtual exponents.*

It is easily checked out that $\rho_1\rho_2\lambda_1\lambda_2\mu_1\mu_2 = 1$, which is a q-analogue of Fuchs' relation for Fuchsian differential equations. If all of exponents are not zero, (6) is written only by exponents:

$$\lambda_1\lambda_2(x - \rho_1)u(xq^2) - \{(\lambda_1 + \lambda_2)x - \lambda_1\lambda_2\rho_1(\mu_1 + \mu_2)\}u(xq) + (x - \lambda_1\lambda_2\mu_1\mu_2\rho_1)u(x) = 0,$$

which is a q-analogue of Papperitz's differential equation [17].

Definition 4.2. *For* (6)*, we set a q-analogue of the Riemann scheme:*

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & \lambda_1 & \rho_1 \\ \mu_2 & \lambda_2 & \rho_2 \end{matrix} \;;\quad x \right\}.$$

The Riemann scheme represents a space of solutions. A q-analogue of the Riemann scheme has already shown by Hahn [4], but Hahn's Riemann scheme is just a table setting out coefficients a_j, b_k. Since exponents is essential in differential or difference equations, we array exponents in the scheme as the same as the original Riemann scheme.

We study transformations which keep hypergeometric type. We consider two difference equations which are transformed by such transformations are equivalent.

Theorem 4.1. *The following four transformations on q-difference equations keep the hypergeometric type. (1) $x \to cx$, (2) $u \to x^\gamma u$, (3) $x \to 1/x$, (4) $u \to (\rho_2 x; q)_\infty/(x/\rho_1 q; q)_\infty u(x)$. By these transformations, the Riemann scheme is transformed as follows:*

(1) $x \to cx$:

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & \lambda_1 & \rho_1 \\ \mu_2 & \lambda_2 & \rho_2 \end{matrix} \;;\quad cx \right\} = \Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & \lambda_1 & \rho_1/c \\ \mu_2 & \lambda_2 & c\rho_2 \end{matrix} \;;\quad x \right\}$$

(2) $u \to x^\gamma u$ for $c = q^\gamma$:

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & \lambda_1 & \rho_1 \\ \mu_2 & \lambda_2 & \rho_2 \end{matrix} \;;\quad x \right\} = x^\gamma \Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1/c & c\lambda_1 & \rho_1 \\ \mu_1/c & c\lambda_2 & \rho_2 \end{matrix} \;;\quad x \right\}$$

(3) $x \to 1/x$:

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & \lambda_1 & \rho_1 \\ \mu_2 & \lambda_2 & \rho_2 \end{matrix} \;;\quad \frac{1}{x} \right\} = \Phi\left\{\begin{matrix} 0 & \infty & * \\ \lambda_1 & \mu_1 & \rho_2 \\ \lambda_2 & \mu_2 & \rho_1 \end{matrix} \;;\quad q^2 x \right\}$$

$$= \Phi\left\{\begin{matrix} 0 & \infty & * \\ \lambda_1 & \mu_1 & \rho_2 q^{-2} \\ \lambda_2 & \mu_2 & \rho_1 q^2 \end{matrix} \;;\quad x \right\}$$

(4) $u \to (\rho_2 x; q)_\infty/(x/\rho_1 q; q)_\infty u$:

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & \lambda_1 & \rho_1 \\ \mu_2 & \lambda_2 & \rho_2 \end{matrix} \;;\quad x \right\} = \frac{(x/\rho_1 q; q)_\infty}{(\rho_2 x; q)_\infty} \Phi\left\{\begin{matrix} 0 & \infty & * \\ \mu_1 & q/\mu_1\mu_2\lambda_1 & 1/\rho_2 q \\ \mu_2 & q/\mu_1\mu_2\lambda_2 & 1/\rho_1 q \end{matrix} \;;\quad x \right\}$$

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Remark. If we apply the transformation (4) to the basic hypergeometric series $_2\varphi_1(a, b; c; q, x)$, we obtain Heine's transformation

$$_2\varphi_1(a, b; c; q, x) = \frac{(abx/c; q)_\infty}{(x; q)_\infty} {}_2\varphi_1\left(\frac{c}{a}, \frac{c}{b}; c; q, \frac{ab}{c}x\right).$$

We consider the transformations not only for individual solutions but also for q-difference equations. Hahn's formula (3) for the q-Bessel function is essentially the same as the transformation (4). When some of the exponents are zero or the infinity, we take $\theta_q(sx)$ instead of $(sx; q)_\infty$ in (4) as we have seen on Lemma 3.1.

By means of transformations in Theorem 4.1, we easily obtain a part of q-analogue of Kummer's twenty-four solutions of the hypergeometric equations.

Proposition 4.2. *The q-difference equation*

$$(c - abqx)u(xq^2) - [c + q - (a + b)qx]u(qx) + q(1 - x)u(x) = 0.$$

for the basic hypergeometric series $_2\varphi_1(a, b; c; q, x)$ have the following eight solutions:

$$_2\varphi_1(a, b; c; q, x), \qquad \frac{(abx/c; q)_\infty}{(x; q)_\infty} {}_2\varphi_1\left(\frac{c}{a}, \frac{c}{b}; c; q, \frac{ab}{c}x\right),$$

$$x^{1-\gamma} {}_2\varphi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; q, x\right), \quad x^{1-\gamma} \frac{(abx/c; q)_\infty}{(x; q)_\infty} {}_2\varphi_1\left(\frac{q}{a}, \frac{q}{b}; \frac{q^2}{c}; q, \frac{ab}{c}x\right),$$

$$x^{-\alpha} {}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right), \quad x^{-\alpha} \frac{(q/x; q)_\infty}{(cq/abx)_\infty} {}_2\varphi_1\left(\frac{q}{b}, \frac{c}{b}; \frac{aq}{b}; q, \frac{q}{x}\right),$$

$$x^{-\beta} {}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right), \quad x^{-\beta} \frac{(q/x; q)_\infty}{(cq/abx; q)_\infty} {}_2\varphi_1\left(\frac{q}{a}, \frac{c}{a}; \frac{bq}{a}; q, \frac{q}{x}\right).$$

Here $a = q^\alpha, b = q^\beta$ and $c = q^\gamma$. These solutions are q-analogue of a part of Kummer's twenty-four solutions [4, 13] and tends to Kummer's original twenty-four solutions when $q \to 1$.

Proof. We can easily obtained all of eight solutions applying the transformations in Theorem 4.1 to $_2\varphi_1(a, b; c; q, x)$. $\square$

Remark. A q-analogue of Kummer's twenty-four solutions are considered by Hahn [4]. Other sixteen solutions are not represented by $_2\varphi_1$. We need $_2\varphi_2$ or $_3\varphi_2$ to represent a q-analogue of Kummer's twenty-four solutions.

4.2 Classification of q-difference equations

By the transformations in Theorem 4.1, we can classify all of q-difference equations (6) of the hypergeometric type. This classification gives a coalescent diagram of q-special functions.

Theorem 4.3. *A q-difference equation* (6) *of the hypergeometric type reduces to one of the following equation by transforms in theorem 4.1.*

1) *When $a_1 a_3 b_1 b_3 \neq 0$, Heine's hypergeometric $_2\varphi_1(a, b; c; q, x)$:*

$$(c - abqx)u(xq^2) - [c + q - (a + b)qx]u(qx) + q(1 - x)u(x) = 0.$$

2) *When $b_3 = 0$, $a_1 a_3 b_1 b_2 \neq 0$, $_1\varphi_1(a; c; q, x)$:*

$$(c - aqx)u(xq^2) - (c + q - qx)u(qx) + qu(x) = 0.$$

3-1) *When $b_1 = b_2 = 0$, $a_3 \cdot a_2 a_1 b_3 \neq 0$, Jackson's Bessel functions $E_\nu^{(1)}(x; q)F$*

$$u(xq^2) - (q^\nu + q^{-\nu})u(xq) + (1 + x/4)u(x) = 0.$$

3-2) *When $b_1 = b_3 = 0$, $a_2 \cdot a_3 a_1 b_2 \neq 0$, Hahn-Exton's Bessel functions $E_\nu^{(3)}(x; q)$:*

$$u(xq^2) + [-(q^\nu + q^{-\nu}) + q^{2-\nu}x]u(xq) + u(x) = 0.$$

3-3) *When $b_3 = a_1 = 0$, $a_2 b_2 \cdot a_3 b_1 \neq 0$, q-Hermite-Weber $_1\varphi_1(a; 0; q, x)$*

$$axu(xq^2) + (1 - x)u(xq) - u(x) = 0.$$

4-1) *When $b_1 = a_2 = b_3 = 0$, q-Airy $_1\varphi_1(0; -q; q, -x)$:*

$$u(xq^2) + xu(xq) - u(x) = 0.$$

4-2) *When $a_1 = b_2 = b_3 = 0$, the Ramanujan function $_0\varphi_1(-; 0; q, -tq)$:*

$$qxu(xq^2) - u(xq) + u(x) = 0.$$

We list up the Riemann scheme corresponding to the classification in the Theorem 4.3. In the following, A and B are quasi-constants, i.e. $A(qx) = A(x), B(qx) = B(x)$.

(1) For $_2\varphi_1(a, b, ; c; q, x)$, the Riemann scheme is given by

$$\Phi \left\{ \begin{array}{ccc c} 0 & \infty & * & \\ 1 & a & c/abq & ; \quad x \\ q/c & b & 1 & \end{array} \right\}.$$

The solution space Φ is

$$\Phi = A\, _2\varphi_1(a, b; c; q, x) + B\, x^{1-\gamma}\, _2\varphi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; q, x\right)$$

around $x = 0$. Here $q^\gamma = c$.

(2) For $_1\varphi_1(a; c; q; x)$, the Riemann scheme is given by

$$\Phi \left\{ \begin{array}{ccc c} 0 & \infty & * & \\ 1 & a & c/aq & ; \quad x \\ q/c & \infty & 0 & \end{array} \right\}.$$

The solution space Φ is

$$\Phi = A\,{}_1\varphi_1(a;c;q,x) + B\,x^{1-\gamma}{}_1\varphi_1\left(\frac{aq}{c};\frac{q^2}{c};q,x\right)$$

around $x = 0$. Here $q^\gamma = c$.

(3-1) The Riemann scheme is given by

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ q^\nu & 0 & \infty \\ q^{-\nu} & 0 & -1/4 \end{matrix}\quad;\quad x\right\}.$$

The solution space Φ is

$$\Phi = A\,x^\nu{}_2\varphi_1(0,0;q^{1+\nu};q,-x/4) + B\,x^{-\nu}{}_2\varphi_1(0,0;q^{1-\nu};p,-x/4)$$

around $x = 0$.

The standard form of Jackson's first q-Bessel function is

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu\,{}_2\varphi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right).$$

Solutions of the q-linear equation

$$u(xq^2) - (q^{\nu/2} + q^{-\nu/2})u(xq) + \left(1 + \frac{x^2}{4}\right)u(x) = 0 \tag{7}$$

is $u = A\,J_\nu^{(1)}(x;q) + B\,J_{-\nu}^{(1)}(x;q)$. The equation (7) is obtained by a shearing transformation of (3-1).

(3-2) The Riemann scheme is given by

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ p^\nu & 0 & \infty \\ p^{-\nu} & \infty & 0 \end{matrix}\quad;\quad x\right\}.$$

The solution space Φ is

$$\Phi = A\,x^\nu\,{}_1\varphi_1\left(0;q^{1+2\nu};q,q^2x\right) + B\,x^{-\nu}\,{}_1\varphi_1\left(0;q^{1-2\nu};q,q^{2-2\nu}x\right).$$

(3-3) The Riemann scheme is given by

$$\Phi\left\{\begin{matrix} 0 & \infty & * \\ 1 & a & 0 \\ \infty & \infty & 0 \end{matrix}\quad;\quad x\right\}.$$

The solution space Φ is

$$\Phi = A\,{}_1\varphi_0(a;-;q,x) + B\,\theta(-ax/q)\,{}_2\varphi_0(q/a,0;-;q,ax/q^2).$$

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(4-1) The Riemann scheme is given by

$$\Phi \left\{ \begin{matrix} 0 & \infty & * \\ 1 & 0 & \infty \\ -1 & \infty & 0 \end{matrix} \quad ; \quad x \right\}.$$

The solution space Φ is

$$\Phi = A \, \mathrm{Ai}_q(x) + B \, e^{\pi i \mathrm{lq}\, x} \mathrm{Ai}_q(-x),$$

where $\mathrm{lq}\, x = \log_e x / \log_e q$.

(4-2) The Riemann scheme is given by

$$\Phi \left\{ \begin{matrix} 0 & \infty & * \\ 1 & \infty & 0 \\ \infty & \infty & 0 \end{matrix} \quad ; \quad x \right\}.$$

The solution space Φ is

$$\Phi = A \, {}_0\varphi_1(-;0;q,-qx) + B \, \theta_q(x) \, {}_2\varphi_0(0,0;-;q,-x/q).$$

The Newton diagram of equations in Theorem 4.3 is as follows. The black circle means a coefficient which is not zero. The Newton diagram of (3-2) and (4-1) are the same, because the q-Airy function is a special case of $E_\nu^{(1)}(x;q)$ in case $q^\nu = \pm i$. The Newton diagram explains that the coalescent diagram in Figure 1. If the Newton diagram of a q-difference equation is a subset of the the Newton diagram of another q-difference equation, we may take a suitable limit.
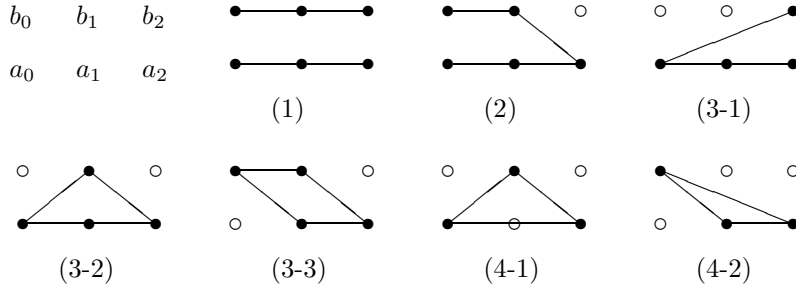

Figure 3: The Newton diagram

Lemma 4.4. *In Theorem 4.3, (4-2) is equivalent to (4-1) by shearing transformation.*

This lemma shows that there exists a relation between the q-Airy function and the Ramanujan function. Recently, Morita [15] has shown a connection formula between $\mathrm{Ai}_q(x)$ and $\mathrm{A}_q(x)$:

$$\mathrm{A}_{q^2}(-q^3/x^2) = \frac{q^2}{(q,-1;q)_\infty} \left(-\theta(-x/q)\mathrm{Ai}_q(xq^2) + \theta(x/q)\mathrm{Ai}_q(-xq^2) \right). \quad (8)$$

5 Hypergeometric solutions of the q-Painlevé equations

As the same as the Painlevé differential equations have particular solutions represented by (confluent) hypergeometric functions, the q-Painlevé equations also have special solutions written by q-hypergeometric functions.

Kajiwara, Masuda, Noumi, Ohta, and Yamada has studied q-hypergeometric solutions of the q-Painlevé equations[9]. The degeneration diagram of q-hypergeometric solutions of the q-Painlevé equations is as follows:

$$q\text{-}\mathbf{P} \quad q\text{-}P_{\mathrm{VI}} \;\rightarrow\; q\text{-}P_{\mathrm{V}} \;\rightarrow\; \begin{array}{c} q\text{-}P_{\mathrm{IV}} \\ q\text{-}P_{\mathrm{III}} \end{array} \quad\rightarrow\quad q\text{-}P_{\mathrm{II}} \quad\rightarrow\quad q\text{-}P_{\mathrm{I}}$$

$$\mathbf{HG} \quad {}_2\varphi_1 \;\rightarrow\; {}_1\varphi_1 \;\rightarrow\; \begin{array}{c} {}_1\varphi_1\,(a;0;q,z) \\ {}_1\varphi_1\,(0;b;q,z) \end{array} \;\rightarrow\; {}_1\varphi_1\,(0;-q;q,z) \;\rightarrow\; \text{none}$$

$$(1) \;\rightarrow\; (2) \;\rightarrow\; \begin{array}{c} (3\text{-}3) \\ (3\text{-}2) \end{array} \;\rightarrow\; (4\text{-}1) \;\rightarrow\; \text{none}$$

Comparing our list in Theorem 4.3, we do not have (3-1) and (4-2). The equation (4-2) is related to (4-1) by a shearing transformation. The equation (3-1) appears in another form of q-P_{III}.

It is known that there are several types of the q-Painlevé equations, whose limit $q \to 1$ tends to the same Painlevé differential equation. It is widely known that there exist two different types of q-P_{III}. One is called q-$P_{\mathrm{III}}(A_5^{(1)})$ by Sakai [20]:

$$\frac{y\bar{y}}{a_3 a_4} = -\frac{\bar{z}(\bar{z} - b_2 t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{y(y - a_1 t)}{a_4(y - a_3)}.$$

Here the affine root system $(A_5^{(1)})$ means the type of the initial value space of the q-Painlevé equation, which is completely classified by Sakai [20].

Another equation is shown by Ramani, Grammaticos and Hietarinta [19]:

$$\frac{\overline{w}\underline{w}}{a_3 a_4} = \frac{(w - a_1 s)(w - s a_2)}{(w - a_1)(w - a_4)}, \tag{9}$$

which is a symmetric specialization of q-P_{VI} found by Jimbo and Sakai [8].

Jimbo and Sakai found q-P_{VI} as a connection preserving deformation of a linear q-difference equation. Their q-$P_{\mathrm{VI}}(A_3^{(1)})$ is given by

$$\frac{y\bar{y}}{a_3 a_4} = \frac{(\bar{z} - b_1 t)(\bar{z} - b_2 t)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \quad \frac{z\bar{z}}{b_3 b_4} = \frac{(y - a_1 t)(y - a_2 t)}{(y - a_3)(y - a_4)}, \quad \left(\frac{b_1 b_2}{b_3 b_4} = q\frac{a_1 a_2}{a_3 a_4}\right).$$

For q-$P_{\mathrm{VI}}(A_3^{(1)})$, we set

$$t = s^2, \; q = p^2, \; b_i = p a_i \;(i = 1, 2), \; b_i = a_i \;(i = 3, 4),$$
$$y(t) = \overline{w}(s), \; z(t) = w(s).$$

Then we obtain the equation (9). In this sense, (9) as can be considered as a symmetric specialization of q-P_{VI}. Since the type of the initial value space of (9) is the same as the type of q-$P_{\text{VI}}(A_3^{(1)})$, we may denote (9) as q-$P_{\text{III}}(A_3^{(1)})$. Kajiwara, Ohta and Satsuma[10] has shown that $J_\nu^{(1)}(x;q)$ is a special solution of q-$P_{\text{III}}(A_3^{(1)})$. This fact is quite natural since $J_\nu^{(1)}(x;q)$ can be represented by a basic hypergeometric series $_2\varphi_1(p^{\nu+1/2}, -p^{\nu+1/2}; p^{2\nu+1}; p, x)$, where $p = \sqrt{q}$.

Theorem 5.1. *In the degeneration scheme of q-special functions in Theorem 4.3, six types of q-special functions appear as special solutions of the q-Painlevé equations. Type (3-1) is appeared as special solutions of a symmetric specialization of q-P_{VI}. Type (4-2) does not appear as special solutions of the q-Painlevé equations, but a shearing transform of (4-2) is equivalent to the type (4-1), which is appeared as special solitons of q-P_{II}.*

6   Matrix Form

In 2×2 matrix form, a q-difference equations of the hypergeometric type is given by
$$Y(xq) = A(x)Y(x) = (A_0 + A_1 x)\, Y(x) \tag{10}$$

Here
$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Matrix forms have the following transformations:
   (1) Change $x \to cx$: $A_0 + A_1 x \to A_0 + cA_1 x$
   (2) Change $Y \to x^{-\gamma}Y$ $(c = q^\gamma)$: $A_0 + A_1 x \to cA_0 + cA_1 x$
   (3) Change $x \to 1/x$ and $Y \to f(x)Y$, where $f(x)$ is a scalar function
   (4) Change $Y \to PY$: $A_0 + A_1 x \to P^{-1}(A_0 + cA_1)P$
The equation (10) has eight parameters A_0 and A_1, but we may generically deduce to three by the transformations (1-4).

Lemma 6.1. *We assume that $b_{12} = 0$ in (10). Then $y_1(x)$ satisfies a single equation of the hypergeometric type.*

Proof. If we eliminate y_2, we obtain

$$y_1(xq^2) = [a_{11} + a_{22} + (b_{11}q + b_{22})x]y_1(xq)$$
$$+ [(a_{12}a_{21} - a_{11}a_{22}q) + (a_{12}b_{21} - a_{11}b_{22} - a_{22}b_{11})x - b_{11}b_{22}x^2]y_1(x) = 0.$$

By Lemma 3.1, this equation reduces to the hypergeometric type. $\qquad\square$

Since we may assume $b_{12} = 0$ by a transformation (4), this assumption is not essential. We give a classification theorem for q-difference equation of 2×2 matrix type:

Theorem 6.2. *A q-difference equation* (10) *reduces to one of the following equation by transforms (1-4) except for the cases (i) A_0 or A_1 is a zero matrix, (ii) $\det A(x) \equiv 0$.*

1) $\det A_0 \neq 0, \det A_1 \neq 0$: $_2\varphi_1(a, b; c; q, x)$

$$A(x) = \begin{pmatrix} 1 & (1-a)/c \\ 0 & q/c \end{pmatrix} + \begin{pmatrix} -b/c & 0 \\ (c-b)q/c & -aq/c \end{pmatrix} x.$$

2) $\det A_0 \neq 0, \det A_1 \sim \mathrm{diag}(0, \mu), \det A(x) \neq const.$: $_1\varphi_1(a; c; q, x)$

$$A(x) = \begin{pmatrix} 1 & (1-a)/c \\ 0 & q/c \end{pmatrix} + \begin{pmatrix} -1/c & 0 \\ -q/c & 0 \end{pmatrix} x.$$

3-1) $\det A_0 \neq 0, A_1^2 = 1$: $E_\nu^{(1)}(x; q)$

$$A(x) = \begin{pmatrix} q^\nu & 1 \\ 0 & q^{-\nu} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1/4 & 0 \end{pmatrix} x.$$

3-2) $\det A_0 \neq 0, A_1 \sim \mathrm{diag}(0, \mu), \det A(x) = const.$: $E_\nu^{(3)}(x; q)$

$$A(x) = \begin{pmatrix} 0 & 1 \\ -1 & q^\nu + q^{-\nu} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -q^{2-\nu} \end{pmatrix} x.$$

3-3) $A_0 \sim \mathrm{diag}(0, \lambda), A_1 \sim \mathrm{diag}(0, \mu)$: $_1\varphi_1(a; 0; q, x)$

$$A(x) = \begin{pmatrix} a & a \\ q - a & q - a \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -q \end{pmatrix} x.$$

4-1) $\det A_0 \neq 0, A_1 \sim \mathrm{diag}(0, \mu), \det A(x) = const.$: *the q-Airy function*

$$A(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x.$$

4-2) $\det A_0 \sim \mathrm{diag}(0, \lambda), A_1^2 = 1$: *the Ramanujan function*

$$A(x) = \begin{pmatrix} 1 & -q^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x.$$

5) $A_0^2 = 1, A_1^2 = 1$:

$$A(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x.$$

Proof. It is easily checked by direct calculation. We show a single equation satisfied by y_1.

1) y_1 satisfies

$$y_1(xq^2) - \frac{c + q - (a+b)qx}{c} y_1(qx) + \frac{q(1-x)(c-abx)}{c^2} y_1(x) = 0.$$

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And $y_1(x) = {}_2\varphi_1(a, b; c; q, x)/(abx/c; q)_\infty$ is a special solution. A fundamental solution is

$$Y = \frac{1}{(abx/c; q)_\infty} \begin{pmatrix} {}_2\varphi_1(a, b; c; x) & \frac{1-a}{q-c}x^{1-\gamma}{}_2\varphi_1(aq/c, bq/c; q^2/c; x) \\ \frac{b-c}{1-c}{}_2\varphi_1(aq, b; cq; x)x & x^{1-\gamma}{}_2\varphi_1(aq/c, b/c; q/c; x) \end{pmatrix}.$$

2) y_1 satisfies

$$y_1(xq^2) - \frac{(c+q-qx)}{c}y_1(qx) + \frac{q(c-ax)}{c^2}y_1(x) = 0,$$

And $y_1(x) = {}_1\varphi_1(a; c; q, x)/(ax/c; q)_\infty$ is a special solution. A fundamental solution is

$$Y = \frac{1}{(ax/c; q)_\infty} \begin{pmatrix} {}_1\varphi_1(a; c; x) & \frac{1-a}{q-c}x^{1-\gamma}{}_2\varphi_1(aq/c; q^2/c; x) \\ \frac{1}{1-c}{}_1\varphi_1(aq; cq; x)x & x^{1-\gamma}{}_2\varphi_1(aq/c; q/c; x) \end{pmatrix}.$$

3-1) y_1 satisfies

$$y_1(xq^2) - (q^\nu + q^{-\nu})y_1(xq) + (1 + x/4)y_1(x) = 0,$$

which is the equation of $E_\nu^{(1)}(x; q)$. A fundamental solution is

$$Y = \begin{pmatrix} x^\nu {}_2\varphi_1(0, 0; q^{2\nu+1}; -x/4) & \frac{1}{q^{-\nu}-q^\nu}x^{-\nu}{}_2\varphi_1(0, 0; q^{-2\nu+1}; -x/4) \\ \frac{q^\nu}{4(1-q^{2\nu+1})}x^{\nu+1}{}_2\varphi_1(0, 0; q^{2\nu+2}; -x/4) & x^{-\nu}{}_2\varphi_1(0, 0; q^{-2\nu}; -x/4) \end{pmatrix}.$$

3-2) y_1 satisfies

$$y_1(xq^2) + [-(q^\nu + q^{-\nu}) + q^{2-\nu}x]y_1(xq) + y_1(x) = 0,$$

which is the equation of $E_\nu^{(3)}(x; q)$. A fundamental solution is

$$Y = \begin{pmatrix} x^\nu {}_1\varphi_1\left(0; q^{1+2\nu}; q; q^2x\right) & x^{-\nu}{}_1\varphi_1\left(0; q^{1-2\nu}; q; q^{2-2\nu}x\right) \\ q^\nu x^\nu {}_1\varphi_1\left(0; q^{1+2\nu}; q; q^3x\right) & q^{-\nu}x^{-\nu}{}_1\varphi_1\left(0; q^{1-2\nu}; q; q^{3-2\nu}x\right) \end{pmatrix}.$$

3-3) y_1 satisfies

$$y_1(xq^2) + q(x-1)y_1(xq) - aqy_1(x) = 0,$$

And $y_1(x) = {}_1\varphi_1(a; 0; q, x)/\theta_q(-ax)$ is a special solution. A fundamental solution is

$$Y = \frac{1}{\theta_q(-ax)} \begin{pmatrix} {}_1\varphi_1(a; 0; q; x) & \frac{a}{q-a}\theta_q(-ax/q){}_2\varphi_0\left(q/a, 0; -; q; ax/q^2\right) \\ {}_1\varphi_1(a/q; 0; q; xq) & \theta_q(-ax/q){}_2\varphi_0\left(q^2/a, 0; -; q; ax/q^2\right) \end{pmatrix}.$$

4-1) y_1 satisfies
$$y_1(xq^2) + xy_1(xq) - y_1(x) = 0,$$

which is the q-Airy equation. A fundamental solution is

$$Y = \begin{pmatrix} {}_1\varphi_1(0; -q; q, -x) & e^{\pi i \mathrm{lq}\, x}\, {}_1\varphi_1(0; -q; q, x) \\ {}_1\varphi_1(0; -q; q, -xq) & -e^{\pi i \mathrm{lq}\, x}\, {}_1\varphi_1(0; -q; q, qx) \end{pmatrix}.$$

4-2) y_1 satisfies

$$y_1(xq^2) - y_1(xq) + x/q\, y_1(x) = 0.$$

And $y_1(x) = \mathrm{A}_q(x)/\theta_q(x/q)$ is a special solution. A fundamental solution is

$$Y = \begin{pmatrix} \frac{1}{\theta_q(x/q)}\, {}_0\varphi_1\left(-; 0; q; -qx\right) & \frac{1}{x}\, {}_2\varphi_0\left(0, 0; -; q; -x/q\right) \\ \frac{q}{\theta_q(x/q)}\, {}_0\varphi_1\left(-; 0; q; -x\right) & {}_2\varphi_0\left(0, 0; -; q; -x/q^2\right) \end{pmatrix}.$$

5) y_1 satisfies

$$y_1(xq^2) - x y_1(x) = 0.$$

And $y_1(x) = 1/\theta_{q^2}(x)$ is a special solution. Since $A(xq)A(x) = \mathrm{diag}(x, xq)$, this case reduces to a q-difference equation of the first order. A fundamental solution is

$$Y = \begin{pmatrix} 1/\theta_{q^2}(x) & \theta_{q^2}(x/q)/\theta_q(x/q) \\ 1/\theta_{q^2}(qx) & \theta_{q^2}(x)/\theta_q(x) \end{pmatrix}.$$

$\square$

7   Summary

There exist seven types of q-hypergeometric equations:

$${}_2\varphi_1,\ {}_1\varphi_1,\ {}_2\varphi_1(0, 0; c),\ {}_1\varphi_1(0; c),\ {}_1\varphi_1(a; 0),\ {}_1\varphi_1(0; -q),\ {}_0\varphi_1(-; 0).$$

Five of seven q-hypergeometric functions correspond to particular solutions of q-Painlevé equations.   Jackson's first q-Bessel ${}_2\varphi_1(0, 0; c)$ corresponds to particular solutions of q-P_{III}, which is a symmetric specialization of q-P_{VI}.

In three types of q-Bessel functions, Hahn-Exton's q-Bessel $J_\nu^{(3)}(x; q)$ might be a right q-Bessel function. In two types of q-Airy functions, Hamamoto-Kajiwara-Witte's q-Airy function $\mathrm{Ai}_q(x)$ might be a right q-Airy function. The Ramanujan function ${}_0\varphi_1(-; 0)$ is connected to the q-Airy function by a shearing transformation and Morita's connection formula.

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